This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

BAKER

ath 5239.05.3

SCIENCE CENTER LIBRARY

SIGILL: COLL: HARVARD: CANTAB: NOV: ANGL: 1650.
CHRISTO ET ECCLESIAE

BOUGHT WITH
THE INCOME FROM
THE BEQUEST OF
NATHANIEL I. BOWDITCH,
OF BOSTON,
(Class of 1822,)

THE ART OF GEOMETRY

A LABORATORY MANUAL FOR STUDENTS' USE

TO ACCOMPANY ANY TEXT-BOOK

BY

ARTHUR LATHAM BAKER, PH.D.
MANUAL TRAINING HIGH SCHOOL, BROOKLYN, N.Y.

SIBLEY & COMPANY
BOSTON CHICAGO

Math 5239.05.3
~~Math~~ 5076.8

Bowditch fund

COPYRIGHT, 1905,
BY ARTHUR LATHAM BAKER.

PREFACE

The pedagogy of Geometry is divisible into three parts, the Science of Geometry (the facts), the Logic of Geometry (the framework), and the Art or Technique of Geometry.

The Science and Logic of Geometry are presented in the current text-books.

The Art of Geometry is given in the following pages.

One of the great causes of complaint among students of Geometry is the lack of a systematic course of procedure. The student has no guide as to *what to do next,* and is generally unable to give any reason for the different steps he takes, except that they are logically correct and that they produce an answer. On the other hand, he ought to know *why* each step in the proof is taken. Only thus can he proceed with real understanding and be prepared to attack intelligently a new and unknown problem. Thus in the problem, To compute the length of the bisector of the angle of a triangle, the first step given by all the text-books is, draw the circumscribing circle. But why? What suggests such a step? What preparation has been given to the student to lead him to such a construction? No text-book has yet answered the question.

The following pages will show not only what suggests the circle, but will provide the student with such a course of procedure (technique) that he could not miss it if he would.

The current text-books supply the student with propositions and a logical proof of their correctness, but they do not put him in a position to deduce his own propositions nor to build a proof independently for himself. He is given the logical reason, but not the purposive, existential reason for the different steps. The bulk of his course consists in checking the correctness of the printed statements, and not in originating and doing for himself. The logical chain merely checks errors; the existential is creative and puts the pupil in control of the process.

The work in the class room is hampered by the fact that the student generally does not clearly realize

What he has to work with, his *tools.*

What can be *done* with them.

What can *not be done* with them.

How they should be *applied.*

How the application should be made *fruitful.*

How the tools should be *selected.*

The following pages supply this information, and put the student in a position to do something more than merely give a memoriter reproduction of the printed page.

It is suggested that the student use the Syllabus of Propositions and its variations. Use several variations, as this will give breadth of view. Make a syllabus from your own text-book, and prove by means of "The Art of Geometry," referring to the text-book only in case of total failure.

CONTENTS

PAGE
INTRODUCTION 1
GEOMETRIC TOOLS 4
SUGGESTIVE COMBINATIONS 7
CONSTRUCTION (DIAGRAMMATIC) TOOLS 7
ART OF DEMONSTRATION 8
ART OF CONSTRUCTION PROBLEMS 15
SYLLABUS OF PROPOSITIONS 38
PLANE GEOMETRY 43

ABBREVIATIONS

The usual abbreviations of the current books.
$\equiv$, coincides with, etc.
Σ, sum, sum of.
The data and parts belonging to the hypothesis are marked in heavy lines, and the parts under investigation are marked with the letters X, Y, etc., the other letters being used for the parts forming the foundation of the argument.

THE ART OF GEOMETRY

INTRODUCTION

1. Geometry (plane) is the science of those metrical elements produced by lines in a plane, viz. the *angle*, the *line segment*, the *bounded surface*, and their invariable accompaniment, the ratio.

2. The *ultimate aim* in every proposition is the comparison (including measurement) of some pair of these elements to show equality or inequality. This dominates the whole course of proof and is the controlling factor in determining the character of the proof, and is therefore called the **Dominant Operation.**

The measurement of one element in terms of others is merely an extension of this thought.

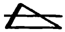
3. The *ascertainment of the Dominant Operation* shows the specific goal which the proposition is aiming at, and prevents waste of time and energy in attempting wrong operations. If comparison of measures is the real operation needed, the student should not waste his time running over and assorting tools (propositions) which have no bearing on that operation.

4. The premises of the hypothesis can be divided into two classes, the unmetrical ones and the *metrical* (measurable) ones. The unmetrical ones have no effect in separating the conclusion of one proposition from that of its neighbors. The controlling factors in producing different conclusions are the metrical premises, and are designated as the **Metrical Hypothesis,** to distinguish from the whole hypothesis.

5. The metrical hypothesis is composed of those metrical facts which distinguish it from its neighbors; those facts which can be stated in the form of an equation; those premises which do not appear in the neighboring propositions. A change in the metrical hypothesis produces a change in the conclusion.

6. The ascertainment of the *metrical hypothesis* shows exactly what and how many metrical facts produce the particular conclusion in question. Illustrations of this are given below.

7.

Unmetrical	Metrical hypothesis	Conclusion
If we have three lines and	one rt. ∠	the other ∠ is not a rt. ∠
	one rt. ∠	the slope line is > the ⊥
	one rt. ∠ and one 0°∠	the other ∠ is a rt. ∠
	one 0°∠	the alt. int. ∠s are =
	alt. int. ∠s =	one ∠ = 0° (lines ∥)
	ext. int. ∠s =	one ∠ = 0° (lines ∥)
	two sides =	two ∠s =
	two rt. ∠s	the other ∠ = 0°
If we have four lines and	one rt. ∠, etc.	the slope lines are =
	two 0°∠s	the opposite segments are =
	the opposite segments =	the lines are ∥
	two = segments, etc.	two other segments =, 
	two pairs of = segments, etc.	a zero ∠

8. The study of geometry has its peculiar tools, as has any other art, and the operations peculiar to those tools: **hand tools,** those things which the hand uses; **brain tools,** those things which the brain uses.

9. The *hand tools* are pencil, paper, chalk, ruler, compass, etc. These produce a *diagram* on paper, etc., which is a correlative[1] of and a symbol for the *mental configuration* with which we are really concerned, the imaginary configuration seen in the mind's eye. With these tools we can perform the *mechanical operations* of *symmetrization* (making the diagram symmetrical, like a butterfly); drawing *unique* (auxiliary) lines, to fit the diagram for a proof; *folding*, in order to bisect an angle or line, or to bring two parts together.

---

[1] Correlatives are those things which suggest and accompany each other, which are indissolubly connected with each other, as father and son, up and down, etc.

10. The **brain tools** are the axioms, theorems, postulates, etc., each theorem becoming a new tool as soon as proved.

With these we are able to perform the **mental operations**: *superposition*, or placing together for the purpose of comparing size; *drawing unique* lines, to prepare the configuration for proof or investigation; *geometric transference*, the carrying of angles, lines, etc., from one part of the configuration to another, for the purpose of addition, comparison, etc., performed by means of the various theorems; and the *logical application* of the brain tools.

11. The technique of geometry consists in the intelligent and purposive use of these tools. The general *course of procedure* is, as in any other art, to select the proper tool for the work in hand, prepare the work for its application, and set the machinery in motion, that is, deduce the consequences, the last consequence being the conclusion or answer sought.

12. This raises the student from being the mere checking clerk of the printed argument into masterful control of the process, guiding it toward the desired goal with the assurance of a master pilot. It transforms him from a mere receptacle of facts into a creative producer.

13. Before the student can attack some of the propositions in geometry with his own initiative, — slope lines cutting off equal segments from the foot of the perpendicular, only one perpendicular, perpendicular is shortest line, slopes cutting off unequal segments from foot of the perpendicular, — he must be provided with a few simple tools in axial symmetry, else he will attack with an empty tool chest.

14. Axial Symmetry is that property possessed by a diagram when, if folded along the axis, all the points come together in pairs, illustrated by folding a piece of paper through a blot of ink.

15. The elements which come together when the diagram is folded are **symmetrical elements.** From these are easily deduced the following theorems:

I. *Symmetrical lines are equal.*
II. *Symmetrical ∠s are equal.*
III. *Symmetrical points lie on the same perpendicular to the axis, and equally distant from it.*
IV. *Symmetrical ∠s on a straight line are right ∠s.*

GEOMETRIC TOOLS[1]

General tools: Falsification of Contradictories. (When applicable.)
Comparison of measures. (When applicable.)

16. COMPARISON OF ANGLES

= ∠s.
Superposition.
Σ ∠s on a st. line = 180°.
Symmetric ∠s are =.
Supplements or complements of = ∠s.
Vertical ∠s are =.
Angles of ||s and transversal.
Base ∠s of isosceles △.
Sides || or ⊥.
Corresponding ∠s in = △s (sim. △s).
∠s in or on the same arc.

---

[1] These tools should be checked by the student as soon as proved and not used till then. They should be numbered as checked, the order varying for different books. Begin the numbering with each subhead.

Axis of kite bisects the ∠s.
Tangents and line of centres.
Comparison of measures.

≠ ∠s. ∠ on broken line is < st. ∠.
Ext. ∠ of △ is < opp. int. ∠.
∠s opp. ≠ sides of same △.
∠s opp. ≠ sides of △s with including sides =, etc.

0° ∠ or ∥s. Coincidence with known ∥.
⊥s to the same line.
∠s of ∥s and transversal =.
Sides of ▱.
Mid line of △.
Intercepts zero arc.

Σ OF ∠s

Σ of measures.
Σ ∠s on st. line = 180°.
Σ ∠s of △ = 180°.

MEASURE OF ∠s

Σ ∠s on st. line = 180°.
Σ ∠s around a point = 360°.
Σ ∠s of a △ = 180°.
Central ∠ is measured by $\frac{1}{2}$ arc, etc.
Any ∠ is measured by $\frac{1}{2}$ sum of arcs, etc.

Right ∠. Superposition on known rt. ∠.
Symmetric ∠s on a st. line.
See measure of ∠s in general.
Median or bisector in isos. △ is ⊥.
Axis and diagonal of kite are ⊥.
Inscribed in semicircle.
Pythagorean Proposition.

0° ∠. See above.

17. COMPARISON OF LINES

= lines. Superposition.
Symmetric lines.
Slope lines cutting = segments, etc.
Corresponding sides in = △s.

Opposite = ∠s in a △.
Transversal cut by equidistant ∥s.
Diagonals of a ▱.
Diagonal of kite bisected by axis.
Radii in a circle.
Tangents to a circle.
Chords equidistant from the centre.
Comparison of measures.

≠ lines. Straight line is shortest.
⊥ is < than any oblique line.
Interior lines < than exterior.
Sum (or dif.) of two sides of a △.
Greater side is opp. the greater ∠.
> side is opp. the > ∠, the including sides =, etc.
Shorter chord is farther from the centre, and *vice versa.*

Measure of lines. Similar △s.
Pythagorean Proposition.
Zero lines. See Concurrency, §§ 42–46.

Coincident lines. ⊥s from the same point.

⊥ lines. See Right ∠.

18. CURVED LINES

= curved lines. Superposition.
= ∠s subtend = arcs.

≠ curved lines. Superposition.
≠ ∠s subtend ≠ arcs.

Measure of. Measure of corresponding central ∠s.[1]

19. COMPARISON OF RATIOS

Comparison of measures.
∥s bisecting (cutting) transversals.
Similar triangles.
Segments of concurrent secants (chords).

20. COMPARISON OF SURFACES

Comparison of measures.
△s (□s) on = bases vary as their alts.
Similar polygons vary as the squares of homol. lines.

---

[1] Used in quadrilateral with opposite supplemental ∠s is inscriptible, etc.

21. SUGGESTIVE COMBINATIONS[1]

Bisected line. Diags. of a ▱.
∥s cutting = segments on a transversal.
∥ bisecting one side of a △.
Diagonal of a kite.
Chord and ⊥ diameter.

Bisected ∠. Median of isos. △.
Axis of kite.
Tangents and centre line.
Equidistance from sides of ∠.

Products: Equal ratios.
Similar △s.
Two intersecting secants (chords).
Transversals cut by ∥s.
: not ratios.
Pythagorean Proposition.
Areas.

Rt. ∠. Inscriptible in a semicircle.

Quad. with opp. ∠s *supplementary.* Inscriptible in circle.

Cross quad. with opp. ∠s *equal.* Inscriptible in circle.

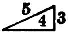
Sim. △s. Sides are proportional.

Ratio. Try composition and division.

22. CONSTRUCTION TOOLS[2]

Rt. ∠.	Median of isos. △.
45° ∠.	Isosceles rt. △.
60° ∠.	Equilateral △.
= ∠s.	= △s.
∥s.	= △s.
0° ∠.	∥s.

Bisection of line. ∥s and transv.; kite; folding; sim. △s.

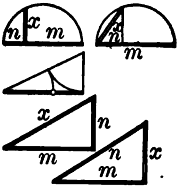
Rt. ∠. Kite; folding; in semicircle; 

½ ∠.	Kite.
½ *arc.*	Kite.
3d, 4th prop.	Similar △s.

---

[1] For use during the demonstration when uncertain what to do next.

[2] Showing what tools are used in the different constructions.



Mean prop.

Medial segment.

$x = \sqrt{m^2 + n^2}$.

$x = \sqrt{n^2 - m^2}$.

$x = \sqrt{mn}$. See Mean proportional.

Inspection will show how few tools are used in making the *constructions* in geometry.

THE ART OF DEMONSTRATION

23. Construct the proper diagram, taking a general form, and avoiding special forms unless they are called for. Draw accurately. An accurate diagram will often *suggest* a solution, as well as *prevent* a false consequence. Do not make the diagram too small. Have *plenty of room* in which to mark results.

24. Mark carefully *on the diagram* the elements which are given by measure; that is, the metrical hypothesis. Mark in heavy lines, etc. This emphasizes the exact foundation of the argument.

25. State the conclusion definitely, in the form of an *equation* where possible. This shows your precise goal and paves the way for the next step.

26. Ascertain the *dominant operation* (so as not to waste your time upon irrelevant matters), whether comparison for equality or inequality or special values, lines equal, unequal, $\angle$s equal or unequal, etc.

The *possible operations* are to show:

Lines, $=$; $\neq$; measure of.
$\angle$s, $=$; $\neq$; measure of, rt. $\angle$s, $\parallel$s; Σ.
Ratios, $=$. (See Appen. A, § 27.)
Surfaces, $=$; measure of (area).

27. Select from the table of tools (§§ 16–20) the *proper tool* and try to apply it, drawing significant lines if necessary. (See Appen. A, § 27.) The general character of the diagram will offer some clew as to the tool. Sometimes several different tools can be selected. (See Appen. A, § 28.)

28. At the beginning of the different parts of the subject, triangles, circles, similar polygons, areas, the number of tools at command is small, and the selection of one which will produce the desired result can be made with comparative ease. In this case the tool selected will dominate the character of the construction necessary to apply it. (See Appen. A, § 28.)

29. Later in the subject, when the tools become more numerous, a trial construction may become necessary before any clew to the tool to be used can be found; but this is exceptional. (See Appen. A, § 29.)

30. If the dominant operation is the *measurement of a line,* then the tool must either be similar triangles (see Appen. A, §§ 28, 33) or the Pythagorean Proposition (or their corollaries), and the investigation then becomes simply a hunt for similar ⚠ or rt. ⚠ (see Appen. A, § 31) and the statement of the proper equations. After that it is algebra, the solution of these equations (see Appen. A, § 30). In either case the equations are liable to become so elaborate as to fall under the head of Algebraic Analysis, the schematization of which is given in § 53.

31. In conclusions involving ratios, or three or more lines, it is sometimes well to consider one as the unknown sought. This makes the diagramming and the algebra simpler. (See Appen. A, §§ 31, 33.)

32. If the dominant operation is the measurement of an angle in terms of others, it is well to turn the theorem into a problem, considering one of the angles unknown and the others as known. (See Appen. A, § 32.)

33. Having drawn the unique lines for the application of the tool, or even in the original diagram, before any auxiliary lines are drawn (see Appen. A, § 33), deduce the *consequences. The consequences should be sought in threefold form:*

1°. **What geometric figure presents itself.**

2°. **What theorems apply to such figures.**

3°. **What is the consequence of the application of those theorems in this case.**

34. The moment a line is drawn note *all the consequences, e.g.* that it gives certain △s, ▱s, etc., that they are rt. △s, that they are similar, and that certain ratios hold between the sides, etc. The omission of a consequence may be of the very one needed.

35. *The triplet of consequences is one of the important features of the investigation.* The question, what propositions apply, will generally narrow the search to a choice of two or three, and *often to a single one.* (See Appen. A, §§ 35, 39.).

Of course, the same diagram will sometimes present several figures, and then each one must be taken up in turn until the process is exhausted or yields a satisfactory answer. The omission to mark some of the consequences is responsible for many failures. The one omitted may have been the very one needed.

36. If the *consequences* fail to give a satisfactory result, add unique (auxiliary) lines so as to introduce into the diagram an additional figure (△, ▱, etc.), to which some of the tools can be applied.

37. Unique lines are those joining specific points in the diagram, or their parallels, bisecting ∠s, etc. It is sometimes well to mark at the outset by *heavy dots* the specific points of the diagram, thus settling at the beginning what unique lines may be drawn. (See Appen. A, § 51.)

38. State the consequences in compact *algebraic equations,* for this purpose indicating the different elements, ⦞, lines, etc., by single letters, and lines and angles which are *equal, by the same letter,* primed and double primed to distinguish them. (See Appen. A, §§ 32, 33, 35, 60.)

39. If several equations result, *operate* upon them *algebraically* (addition, subtraction, etc.) for new results which may have a geometric interpretation. (See Appen. A, §§ 32, 33, 53.) The change of form of a single equation may be suggestive. (See Appen. A, § 39.)

40. Vary the diagram by putting one of the points, circles, etc., in a different position, inside instead of outside, etc. You may get a diagram very much *easier* to work with. (See Appen. A, §§ 40, 51.)

CONCURRENCY THEOREMS

41. The dominant operation is really the measurement of the sides of the triangle formed by the concurring lines, these sides being zero if the lines concur. This method is generally replaced by special methods.

42. A zero segment is shown by the coincidence of two rays, one to each of its ends.

Special Methods

43. Show that the intersection of two of the lines lies on a locus which coincides with the third line, *e.g.* transversals cutting proportional segments on ‖s; angle bisectors of a △; perpendicular bisectors of the sides of a △; altitudes of a △.

44. Connect two points of one of the lines with the intersection of the other two, and show that the two connecting segments form a straight line. (See Appen. A, § 44.)

45. Use the converse of: If three lines through the vertices of a triangle are concurrent, then the products of the three alternate segments of the sides are equal. (See Appen. A, § 45.)

46. Show that the segments of line 1 from a fixed point to its intersections with lines 2 and 3 are the same. (See Appen. A, § 46.)

COLLINEARITY THEOREMS

47. Show that the line segments determined by the three points make an angle of zero or 180°. (See Appen. A, § 47.)

48. Use the converse of: If a transversal cuts the sides of a triangle, the products of the three alternate segments in order are equal. (See Appen. A, § 48.)

SUGGESTIVE QUESTIONS

What is the metrical hypothesis;[1] how many and what facts?

What is the metrical conclusion?

What is the dominant operation; comparison for equality, inequality, or measurement; and of what?

What tool do you select for this operation?

Is any construction needed in order to apply this tool, and if so what?

If you are unable to select a tool, what unique lines will you draw in order to get a suggestion? (Consult Suggestive Combinations, § 21.)

What consequences follow from the nature of the diagram, either before or after the construction; what figures, what facts, what results from the facts?

---

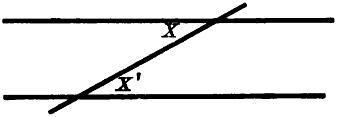
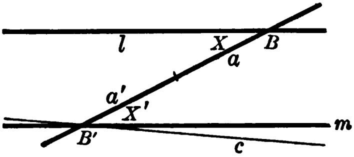
[1] In some cases (Props. No. 1, 17, etc., Appen. B) we must fall back on the straightness of the lines (curvature = zero) for the metrical hypothesis; or the concurrency of lines (zero segments) as in the case of inscribed polygons.

If the consequences are in several equations, what results do you get by operating on these equations by addition, subtraction, etc.? (See § 38.)

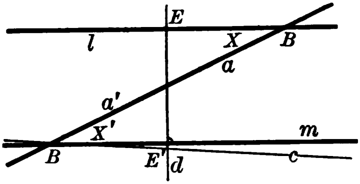
State the proposition in algebraic form, and see what suggestions you can get from the simplified or modified forms of this equation. (See § 39.)

ILLUSTRATIVE DEMONSTRATIONS

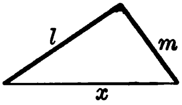
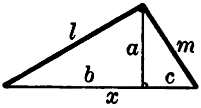
Theorem: *The alternate interior angles of two parallels cut by a transversal are equal.*

How many and what metric facts are found in the hypothesis?	One, a zero ∠; *i.e.* ‖s. 
What is the metric conclusion?	$X = X'$
What is the dominant operation?	Comparison of ∠s; for equality.
What is the tool selected for this operation?	Superposition; by revolving the upper half around on to the lower half.
What construction is needed to accomplish this?	Bisection of the transversal, and revolution; $a = a'$. 
What are the consequences?	$B \equiv B'$; $l \equiv c$, some line above, or below or on m; $a \equiv a'$.

If c is on m, as seems probable, the $\angle\, cm$ must be zero. And now we have an entirely new operation, the measurement of a zero ∠.

The tool for this is, ⊥s from the same point.	
The construction is	the line $d \perp m$, to secure one ⊥ at least.
What consequences?	$E = 90°$ (⊥ to one of two ∥s, etc.) $E' = 90°$(revolved position of E) $\angle cm = 0°$(⊥s from the same point ≡) $\therefore c \equiv m$ $\therefore X = X'$ (sides ≡) Q.E.D.

Theorem: *The square of the hypotenuse is equal to the sum of the squares of the legs.*

How many and what facts constitute the hypothesis?	One; a right angle. 
What is the metrical conclusion?	$x^2 = l^2 + m^2$.
What is the dominant operation?	Measurement of a line, x.
What tool is selected?	Similar triangles, the only one in hand at this stage.
What construction is made to supply the similar ⊿s?	Draw a, ⊥ to x. 

What consequences?	After the ⚠ have been proved similar, we get the nine equations, $\frac{l}{x}=\frac{b}{l}=\frac{a}{m}$ $\frac{l}{a}=\frac{m}{c}=\frac{x}{m}$ $\frac{l}{m}=\frac{a}{c}=\frac{b}{a}$
What results from these equations by algebraic operations?	Picking[1] out the equations which have some peculiarity, the same letter twice, and simplifying, we get, $l^2=bx$ $m^2=cx$ $a^2=bc$ Adding the first two we get, $l^2+m^2=x(b+c)=x^2$ Q.E.D.

THE ART OF CONSTRUCTION PROBLEMS

49. Line Segments.

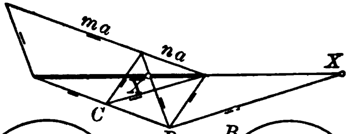
Sum, difference, multiple, submultiple. See Geometry.

Third proportional.
Fourth proportional.
Mean proportional, $x=\sqrt{mn}$.
$x=\sqrt{m^2+n^2}$.
$x=\sqrt{m^2-n^2}$.
Medial segment.
See § 22.

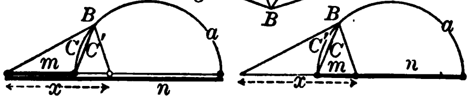
---

[1] We might have simply said, solving for x.

Harmonic segments



Harmonic mean



N.B. The order of the letters is the order of the operations. Note the order from C to C'.

50. Construction of a Whole Configuration from Some of its Parts. The difficulty of this problem consists in getting the different data *into position.* This once accomplished, the whole can be easily constructed.

The order of procedure is as follows:

Analysis: Required whole (in dotted lines and with unknown points indicated by hollow dots) to show exactly what is sought.

Given parts (lines and *two* points heavy) in order to show exactly what points are fixed and known.

Tie lines, $\triangle$s, theorems, to connect the known points with the unknown.

Consequences.

Trial Diagram.

Construction: Reverse the chain of consequences, beginning with any set of points, generally two, which can be fixed. No more than two can be fixed at the outset.

Construction Diagram. (See Appen. A, § 50.)

Sometimes the methods of the sections following will be more productive.

CONSTRUCTION OF CONFIGURATIONS UNDER GIVEN CONDITIONS

Here we have several methods.

(1) Geometric Analysis.
(2) Intersections of Loci.

(3) Algebraic Analysis.
(4) Similar Figures.
(5) Indirect Construction.
(6) Bisection, Trisection of Required Lines.

51. Geometric Analysis.

Assumed relation: Given parts heavy, including heavy dots for the fixed points; required parts dotted, including hollow dots for required points.

Tie lines, $\triangle$s, theorems, to bind the two together.

Consequences.

Analysis Diagram.

Construction: Reversal of analysis, beginning with the known points.

Construction Diagram. (See Appen. A, § 51.)

52. Intersections of Loci.

Construct locus under 1st, 2d, ⋯, conditions.

Locate intersections.

In problems requiring the construction of a circle under given conditions, the first requisite is the centre of the circle. This is found by the intersection of loci.

53. Algebraic Analysis. It is often possible to state the imposed conditions in the form of equations. The solution of the equations gives the required line as the equivalent of an algebraic expression, the geometric interpretation of which will give the required construction line. The schematization is:

Assumed construction.
Trial construction for tie lines.
Equations connecting known and unknown.
Solution of equations for the unknowns.
Geometric interpretation by § 49. (See Appen. A, § 53.)

N.B. Some problems can be solved only by Algebraic Analysis. (See Appen. A, § 53.)

54. Similar Figures. The order of procedure is:

Ratio of similitude determined.

Coincident homologous points assumed (vertex of rays).

Pair of homologous lines located (along the same ray or otherwise).

Remaining homologous lines constructed. (See Appen. A, § 54.)

55. Indirect (Reverse) Construction.

Where we know the form of the desired construction, but not its location, it is generally easier to reverse the construction, taking the required parts for given and the given parts for required. This gives us a diagram similar to the required one, but on a different scale. It only remains to copy this on to the given lines to have a solution of the problem. The scheme of procedure is:

Reversal of conditions (interchange of given and required parts, using diagram of any convenient size).

Copying this diagram on to the original. By any method; conveniently by similar figures. (See Appen. A, § 55.)

56. Bisection of Required Lines by Given Points.

Make the given point the centre of similitude, and complete (partial symmetrization will often suffice) the symmetry of the diagram. The intersection of the given and construction lines will be points on the desired construction.

57. Trisection can be performed by extension of the thought, using opposite perspective with the proper ratio of similitude. (See Appen. A, § 57.)

58. Construction of Loci. In Elementary Geometry the only loci are straight lines and circles, and the general rules for determining loci can be modified as follows:

59. The locus of a point is a *straight line*, when the point is either:

(*a*) At a constant distance from a fixed line.

(*b*) At a constant ∠ from a fixed point on a fixed straight line.

60. The locus is a *circle* when the point is either:

(*c*) At a constant distance from a fixed point.

(*d*) At the vertex of a constant ∠ whose arms pass through fixed points.

The schematization is:

Location of given elements.
Construction of a few points to get a clew.
Selection of type point for discussion.
Tie lines, etc., seeking a *constant ∠ or distance.*
Consequences. (See Appen. A, § 60.)

In some cases *reductio ad absurdum* is easier, *e.g.* the locus of a point from which the tangents to two intersecting circles are equal.

Special cases are inverses of circles and lines.

61. Transformation of Areas.

This operation is based upon the propositions:

Triangles with the same base and altitude are equal.

Triangles, ▱s, with the same ∠ are proportional to the products of the including sides.

Similar polygons are proportional to the squares of homologous lines.

Area of a △ = etc.

Area of a ▱ = etc.

The use of the first one is illustrated in all geometries, the transformation of a polygon into a triangle.

The rest are utilized by stating them in the form of equations and solving for the unknown elements. (See Appen. A, § 61.)

ADDITION OF SURFACES

62. (*a*) Adjoin and obliterate the common boundary, and then reduce by § 61.

(*b*) Reduce to △s (▱s) of the same base, and bring the bases together, and then swing the adjoining sides into one straight line. (See Appen. A, § 62.)

(*c*) Pythagorean Proposition, for similar surfaces.

RATIOS

63. To convert a line ratio $l:m$ into a surface ratio: Construct similar polygons on l and $\sqrt{lm}$ as homologous sides.

64. To convert surface ratio into line ratio: (*a*) Transform the surfaces to △s on the same base and compare the altitudes. (*b*) Use, similar areas upon the sides of a rt. △ are proportional to the projections of those sides upon the hypotenuse.

65. To transform a line ratio into one with a given antecedent or consequent, use sim. △s.

66. Combination of ratios: *Addition*, *subtraction*, and *division*, reduce to the same consequent and combine the antecedents; *multiplication*, reduce to the same consequent for one ratio and antecedent for the other, and select the unlike antecedent and consequent.

NOTATION

67. The technique is very much aided by a systematized notation, the only really new features of which are §§ 64–68, and the systematic and consistent use of the other features, which ordinarily are used spasmodically and at haphazard.

68. Middle letters (l, m, N, $\cdots$) for those parts spoken of in the hypothesis.

Latter letters ($x, y, z, \cdots$) for those parts spoken of in the conclusion.

First letters ($a, B, c, \cdots$) for those parts introduced by the student for the purposes of proof, *exactly in the order of introduction.*

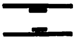
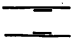
69. Small (lower case) letters for *lines.* Capital letters for everything else, angles, points, surfaces, etc.

70. Heavy lines for the metrical elements of the hypothesis, and heavy dots (●), for the known, fixed points of the diagram. All other lines light. Unknown, sought-for points, hollow dots (○).

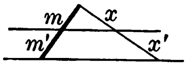
71. Equality of parts indicated by the same letter, primed, double primed, etc.

72. Right ∠ indicated by , , according as it belongs to the hypothesis or not.

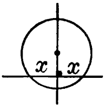
73. Required, sought-for construction lines, doubled lines. Information lines, not belonging to the diagram proper, but used for information only, dotted lines.

Parallel lines by  or .

74. *Advantages.* It emphasizes the metrical hypothesis and accentuates facts otherwise overlooked. Thus



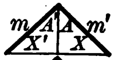
that the length of the base is not an element of the discussion; that the conclusion depends upon two metrical facts, equal segments, parallel lines. So also



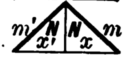
that the length of the radius or diameter is irrelevant, and that the conclusion is produced by two metrical facts, a central line and a rt. ∠.

It distinguishes the diagrams of different propositions. Thus:

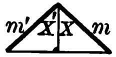
The base ∠s of an isos. △ are =.



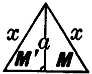
The bisector of the vert. ∠ of an isos. △ bisects the base.



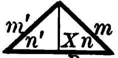
A perpendicular through the vertex bisects the ∠.



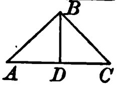
The sides of an isogonal △ are =.



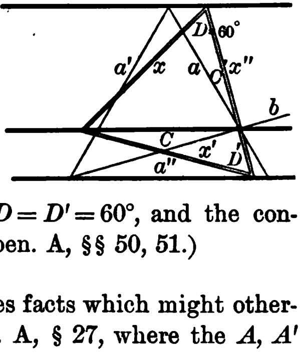
The median of an isos. △ is ⊥ to the base.



Under the current notation the individuality would be lost in the one diagram.



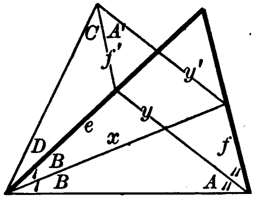
75. It makes the order of procedure particularly easy to follow. Thus the accompanying diagram states the data, the problem, and gives the steps of the solution: To construct an equilateral △ (double lines, $x = x' = x''$) with its vertices on three given ||s (heavy lines), construct an equilateral △ as shown $(a = a' = a'')$, draw the line b, construct the ∠ $C' = C$, lay off $D = D' = 60°$, and the construction is made. (See also Appen. A, §§ 50, 51.)



76. It visualizes and emphasizes facts which might otherwise be overlooked. See Appen. A, § 27, where the A, A' keeps the equality clearly in view; § 28, where the x, x' keeps the goal more vividly in sight; § 30, where the O, O' are suggestive of the similarity of the △s; § 32, where the Y, Y' make the equalities of the ∠s more patent to the eye; § 33, where △ CB, △ $C'B'$ shows the similarity at sight; § 60, where the A, A', a, a' almost force the consequences into view.

APPENDIX A[1]

27. *If a triangle is not isosceles, the bisectors of the base angles are unequal.*



N.B. The equal $\measuredangle$s are indicated by check marks.

The *dominant operation* is comparison of lines to show inequality. Running over the list of *tools*, the most applicable one seems to be: the greater side is opposite the greater angle, — which immediately suggests sliding y into the position y', so as to get the lines into position to form a $\triangle$. Now follow the

Consequences: $A' > B$ (hypoth.)

$C > D$ ($e > f'$, $>$ side opp. $> \angle$)

$\therefore A' + C > B + D$ (addition)

$\therefore x > y'$ ($>$ side opp. $> \angle$) Q.E.D.

Central angles have the same ratio as their intercepted arcs. Here the *dominant operation* is comparison of ratios, and the choice of *tools* is easily made, comparison of measures. Hence the *construction* is to apply the common measure to both angles and arcs, in order, by counting, to get the measures of the ratios, so that they can be compared. An analogy is the comparing of two sand-heaps by counting the bucketfuls in each.

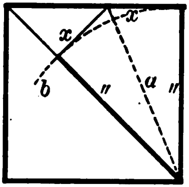
Notice how the selection of the tool almost drives the student to the proper construction.

The same method of procedure applies to, a triangle cut by a line ∥ to the base, rectangles upon the same base.

---

[1] The order of these examples is not intended to be progressive, but merely to illustrate the methods of the Art of Demonstration. Many of the propositions cited as authorities must be taken for granted by the student reading this for the first time. A progressive order of propositions is given in the Syllabus of Propositions. Consult any geometry for diagrams not given here.

28. *On the diagonal of a square is laid off one side, and a perpendicular erected. To prove* $x = x'$.



N.B. The equal sides are indicated by check marks.

If we choose superposition as the tool, then the *construction* is: fold, etc. If corresponding sides of = △s, then the *construction* is: draw the line *a*. If tangents from a point without, then the *construction* is: draw the arc *b*.

Find the length of the side of a regular inscribed 8-gon.

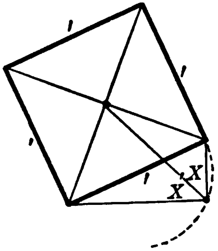
Here we have the two tools, rt. △s, or sim. △s (if we know the circumscribed 8-gon), and the corollary of these, the length of the chord of one-half an arc, if it has been provided.

Ascertain the dominant operation and tool for the propositions mentioned in § 13, viz. symmetric lines, symmetric angles, or superposition.

Ditto for: A straight line ⊥ to one of two ∥s is ⊥ to the other; viz. dominant operation: measure of a rt. ∠; Tool: superposition on known rt. ∠.

Ditto for: Two △s are equal if three sides of one are = respectively to the three sides of the other; viz. dominant operation: comparison of ∠s; Tool: base ∠s of isos. △s, symmetric ∠s.

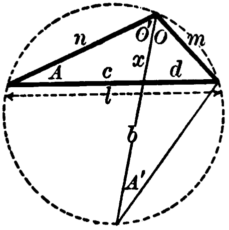
29. *The line joining the centre of the square upon the hypothenuse of a right triangle, to the vertex of the right angle, bisects the right angle.*



N.B. The equal sides are indicated by the check marks.

Here the selection is not clear until the "Suggestive Combinations," § 21, suggests that right angles are inscriptible in a semicircle, whence the circle and the *tool*, ∠s in or on the same arc.

30. *Required the length of the bisector of an angle of a triangle, in terms of the sides.*



The *dominant operation* is the measurement of a line, x, for which there are but two tools, sim. △ and rt. △. The equality of the angles O, O' rather suggests sim. △. Hence the hunt now becomes one for sim. △, which are secured by the *construction*, make $A' = A$, giving the segments b, c, d, as shown. From this we have the

Consequences: $\frac{n}{x} = \frac{x+b}{m}$ (sim. △)

Whence $nm = x^2 + xb$

But this has introduced the unknown element b. To get rid of it, we utilize the suggestive combination, § 21, products, which suggests circumscribing the △ with a circle. It will pass through the vertex of A'. (*Reductio ad absurdum.*)

Whence $nm = x^2 + cd$

To eliminate cd, we have $c + d = l$ (hypoth.)

$\frac{c}{n} = \frac{d}{m}$ (bisector of ∠ of △, etc.)

Whence $cd = \frac{mnl^2}{(m+n)^2}$

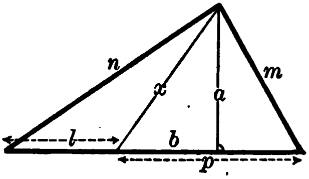
and $x^2 = mn - \frac{mnl^2}{(m+n)^2}$

Notice how each step, from the tool onward, suggests the next one.

Had we chosen the Pythagorean Proposition as the tool, the problem would have been a special case of the following one.

Required the length of a line from the vertex to a known point on the base.

The *dominant operation* is the measurement of a line, and in this case the choice of *tool* must be rt. △s. Hence



Construction: Draw the $\perp$ a in order to get a rt. △. This gives us three rt. △s, for *each one of which* we state the Pythagorean Proposition, viz.:

Consequences:

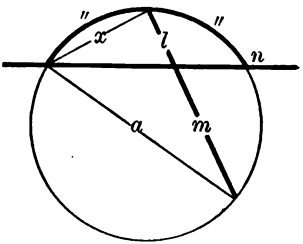
$$n^2 = a^2 + (l + b)^2$$
$$m^2 = a^2 + (p - b)^2$$
$$x^2 = a^2 + b^2$$

Whence it is merely a question of algebra to get

$$x^2 = \frac{lm^2 - lp^2 + pn^2 - pl^2}{l + p}$$

Similarly, in most problems requiring the length of lines, the *geometry is very simple*, the statement of an equation for *each right-angled* triangle; the rest is algebra. This makes the geometry of these problems simply a question of writing a few equations, all of the same type, one for each hypotenuse, or one for each pair of sim. △s. (See Appen. A, §§ 31, 33, 51, 53.)

31. *The chord of half an arc is a mean proportional between a chord from the middle point and the portion of it between the mid point and the chord of the whole arc*, i.e. $x^2 = l\,(m + l)$.

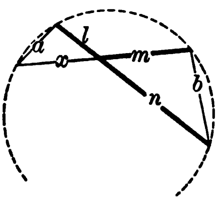


N.B. The equal arcs are shown by the check marks.

The *dominant operation* is the measurement of a line, or the comparison of ratios, and in either case the tool is sim. △s, and the hunt becomes one for sim. $\triangle$s; that is, since one $\angle$, xl, is common, to prove $\angle am = \angle xn$.

Notice how the goal suggests the tool, and how the hunt narrows down to the comparison of two $\angle$s, each step suggesting the next one; especially how the ratio $x : l = (m + l) : x$ points out the very $\triangle$s which must be examined for similarity.

The product of the segments of one of two intersecting chords equals the product of the segments of the other chord.



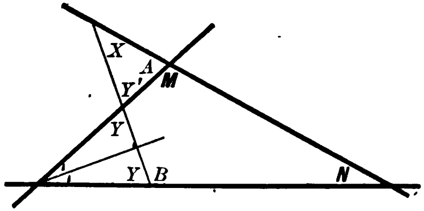
The *dominant operation* being the measurement of a line, the *tool* is sim. $\triangle$s, which immediately suggests the lines a, b (or analogous ones) and the testing of the $\triangle$s for similarity, whence immediately (see "Suggestive Combinations," § 21):

$$x : l = n : m$$

If the student does anything at all in the way of drawing lines, he must get the $\triangle$s, and if he knows his goal, he cannot fail to know what to do with them.

Notice how each step suggests the next one, from the dominant operation on.

32. *The perpendicular to the bisector of an angle of a triangle makes angles with the sides equal to one-half the sum and one-half the difference of the other angles of the triangle.*



The *dominant operation* is the measurement of an angle, and therefore the *tool* is $\Sigma \angle$s of $\triangle = 180°$ Whence the

Consequences: $X + Y' + A = 180°$

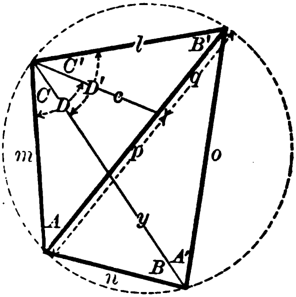
$$X + B + N = 180°$$

$$\left.\begin{array}{l} A = 180^\circ - \boldsymbol{M}, \\ B = 180^\circ - Y, \end{array}\right\} \text{ in order to eliminate } A,\ B.$$

Whence $$X = \frac{\boldsymbol{M} - \boldsymbol{N}}{2}, \quad Y = \frac{\boldsymbol{M} + \boldsymbol{N}}{2}$$

Notice that there are only two $\triangle$ containing the letters X, Y, $\boldsymbol{M}$, $\boldsymbol{N}$, and it is difficult to avoid getting the proper equations, either for the $\triangle$, or for the elimination of A and B.

33. *In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.*



In accordance with the suggestion of § 31 we mark one line y as unknown, and the conclusion is $y(p+q) = ln + mo$.

The *dominant operation* is the measurement of a line y, and therefore the *tool* must be sim. $\triangle$ or rt. $\triangle$, the equality of the angles A, A' leaning toward sim. $\triangle$. Hence the immediate quest is for sim. $\triangle$.

Marking the immediate *consequences* (§ 33) on the diagram as it stands, we have:

$$A = A',\ B = B' \qquad (= \text{arcs})$$

These (since we are hunting for sim. $\triangle$) suggest making $C' = C$ by drawing the line c (or an analogous operation from the vertex of B'), which immediately gives the

Consequence: $D = D'$

$$\triangle BC \text{ sim. } \triangle B'C'$$

$$\triangle AD \text{ sim. } \triangle A'D'$$

Stating the equations (see "Suggestive Combinations," § 21), we have:

$$\frac{l}{c}=\frac{y}{m}$$

$$\frac{l}{o}=\frac{c}{p}$$

$$\frac{m}{n}=\frac{c}{q}$$

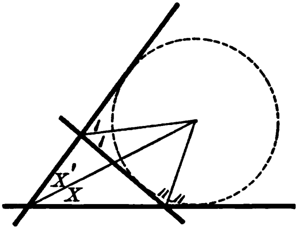
Whence, eliminating c, we get:

$$p=\frac{mo}{y},\quad q=\frac{ln}{y}$$

Whence $$y(p+q)=ln+mo$$

Notice that the mere marking of the consequences A, A', B, B almost forces the construction of the $\angle C'$, *knowing the quest to be for sim.* △.

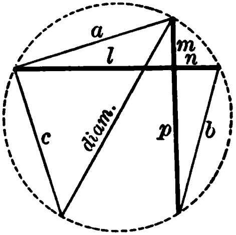
35. *If two exterior angles of a triangle are bisected, the line from the intersection of the bisectors to the opposite vertex bisects the angle.*



The bisectors suggest (see "Suggestive Combinations," § 21) equidistance from the sides, and therefore the tangent circle, and this in turn suggests *tangents*, and this in turn *tangents* and *centre line*, and we have run the chain of consequences down to a single proposition, which can be found by any one who can read.

Notice how each step suggests the next one, leading to the proper tool with almost infallible certainty.

39. *The sum of the circles upon the segments of two perpendicular chords as diameters equals the original circle.*



Stating the theorem algebraically, we have:

$$\pi\left(\frac{l}{2}\right)^2+\pi\left(\frac{m}{2}\right)^2+\pi\left(\frac{n}{2}\right)^2+\pi\left(\frac{p}{2}\right)^2=\pi\,\overline{\text{rad.}}^2$$

$$\because\ l^2+m^2+n^2+p^2=4\,\overline{\text{rad.}}^2=\overline{\text{diam.}}^2$$

But $l^2+m^2=a^2$, $n^2+p^2=b^2$ (see § 33)

$$\because\ a^2+b^2=\overline{\text{diam.}}^2$$

provided the original equation is true. But $a^2+b^2=\overline{\text{diam.}}^2$ rather suggests a rt. △, and therefore, putting the diam. into position from one end of a, we have:

$$a^2+c^2=\overline{\text{diam.}}^2$$

Now, if $b=c$, the two equations become identical, and the original equation is proved true; and the demonstration has narrowed down to the equality of the two lines b, c, easily shown by folding b on to c, or by completing the rt. △ of which c is the hypotenuse.

40. *Through a point near the corner of a field to run a straight fence, cutting off a given area.*

The diagram is much easier to "see through" if the point is within the field, after which the diagram with the point outside the field can be attacked by analogy.

44. If diameters be drawn through one of the points of intersection of two circles, the other ends of the diameters will be on a line concurrent with the other intersection of the circles.

If a square be inscribed in a square, the two sets of diagonals concur.

45. The medians of a triangle concur.

The angle bisectors of a triangle concur.

The altitudes of a triangle concur.

Lines from the vertices of a triangle to the points of tangency of the inscribed circle concur.

46. If three circles intersect, their common chords are concurrent.

The lines from the vertices of a triangle to the points of tangency of the inscribed circle are concurrent.

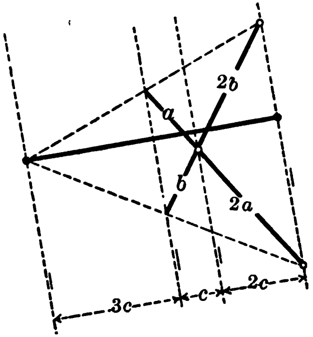
The medians of a triangle meet in a trisection point.

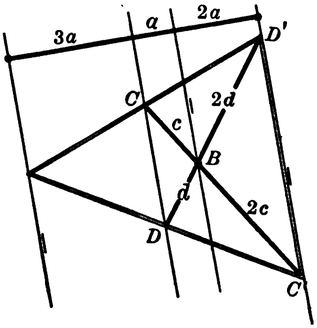
47. If a circle be circumscribed about a triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides are collinear.

48. If a circle be circumscribed about a triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides are collinear.

Tangents at the vertices of an inscribed triangle meet the sides in collinear points.

50. *To construct a triangle, having two medians and the altitude from the third vertex.*





Analysis diagram,

showing the required whole in dotted lines and the consequences a, $2a$, etc.

a, $2a$, b, $2b$. (Medians trisect each other.)

c, $2c$, $3c$. (Def. of median, etc., ∥s cutting transv., etc.)

As soon as we strike the combination $3c$, c, $2c$ we have the key to the position, and it merely remains to reverse the chain, as indicated in the construction diagram.

Construction diagram.

B taken at random on the ∥.

$c = \frac{1}{3}$ one median, giving C, C'.

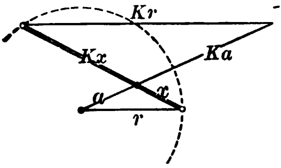
$d = \frac{1}{3}$ second median, giving D, D'.

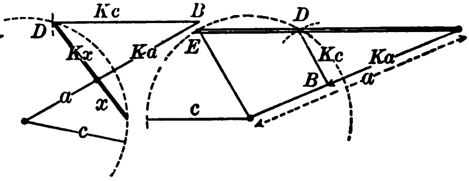
Whence, etc.

Note the order of the letters as significant of the order of operations.

Notice the *positions* of only *two* points are known. The ends of any known line can be assumed as the *known points*, but a proper choice greatly facilitates the "tieing" with the unknown points.

51. *To draw a secant through a given point so that the point will give segments in a given ratio, $1 : K$.*





Analysis diagram,

showing the assumed construction and the known (assumed) points *heavy*.

Construction diagrams.

The order of the letters shows the successive steps.

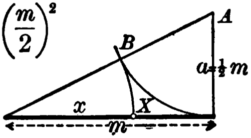
The dominant operation is in effect the measurement of a line, $Kx = K \cdot x$. Hence, the search becomes one for sim. △s, as shown. They could not possibly be avoided (or an analogous pair) if any lines at all are drawn through the marked points.

Reversal of the diagram gives the construction diagram.

53. *To divide a line in extreme and mean ratio.*

$$x^2 = m(m - x). \qquad \text{(Def.)}$$

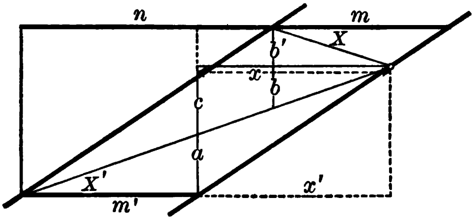
$$\therefore \; x = -\frac{m}{2} + \sqrt{m^2 + \left(\frac{m}{2}\right)^2}$$



Whence, § 49, the diagram annexed.

External division is similarly secured.

From the ends of the base of a parallelogram to draw two lines to a point in the opposite base and making equal angles with the adjacent sides.



This example is soluble *only* by Algebraic Analysis.

The *dominant operation* is measurement of a line x, and therefore the *tool* is sim. △s.

Whence (1) $\dfrac{m}{x} = \dfrac{a}{c}$

(2) $\dfrac{b}{c} = \dfrac{x - n + m'}{x}$

(3) $\dfrac{n}{x'} = \dfrac{c + b + a}{c + a}$

$$\therefore\ (4)\ \frac{m}{m+x}=\frac{a}{c+a} \qquad (1)$$

$$(5)\ \frac{b}{c+a}=\frac{n-x}{x} \qquad (3)$$

$$\frac{a}{b}=\frac{mx}{(m+x)(n-x)} \qquad \frac{(4)}{(5)}$$

$$\frac{a}{b}=\frac{mx}{x^2-nx+mx} \qquad \frac{(1)}{(2)}$$

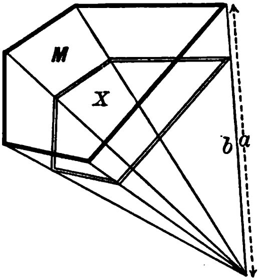
$$\therefore\ x=\frac{n-m}{2}+\frac{\sqrt{n^2+m^2}}{2}$$

Whence x can be constructed by § 49 and the point found.

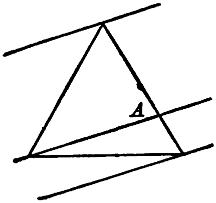
Notice that the quest for sim. △ forces the equations (1), (2), (3), whose solution is algebra, not geometry.

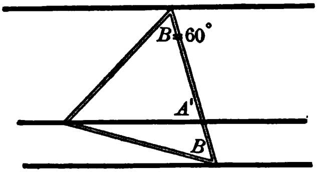
54. *To construct a polygon X similar to* ***M*** *and equivalent to* ***N***.

The first requisite is the ratio of *similitude*, which is determined by reducing both **M** and **N** to similar polygons (squares, or, preferably, similar isosceles △) with the homologous sides a, b. Set these off as rays to any point, and construct the ∥ homologous lines.



55. *To construct an equilateral triangle with its vertices on three given parallels.*





Reversed diagram (any size), with the distance between the ∥s proportional to that of the given ∥s, found by cutting the side at A in the same proportion.

Copy of reversed diagram on to given ∥s, by making $A' = A$, $B = B' = 60°$.

PROBLEMS SOLUBLE BY THE INDIRECT METHOD

Construct an equilat. △, given the sum (or difference) of the side and altitude.

Construct a rt. △, having an ∠ and the radius of the inscribed circle.

Inscribe a circle in a given sector.

Construct a circle of given radius and tangent to two given lines.

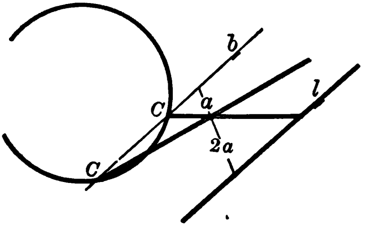
Across a given circle and a tangent line throw parallel to a given line a secant whose segments shall have a given ratio.

In a given triangle to inscribe parallelograms of a given shape (square, rhombus).

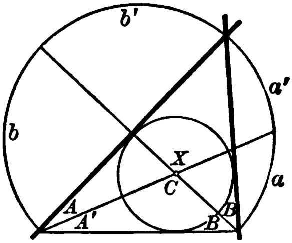
etc. etc. etc.

57. *To draw a line between a circle and a line, which shall be trisected by a given point.*

Here b is the opposite perspective of the given line, thus determining C and the required line. The opposite perspective of l is taken as being easier to draw than that of the circle.



60. *What is the locus of the centre of the inscribed circle of a triangle whose base and vertical angle are constant?*



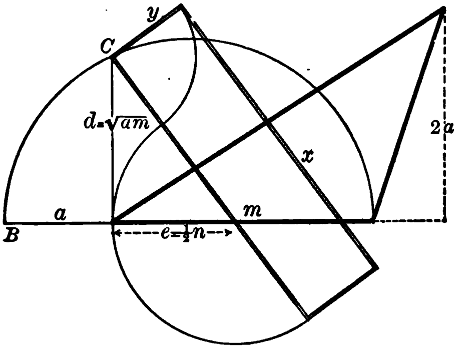
A few trial points indicate a circle and therefore the hunt becomes one for a constant angle, since there is evidently no indicated fixed point.

Assuming a type point, and drawing significant lines as shown, we get the consequences

$$\text{Meas. } (A+B)=\frac{a+b}{2}=\frac{2(a+b)}{4}=\text{const.}$$

Whence $C=180-(A+B)=\text{const.}$, and the locus is a circle.

61. *To transform a triangle into a rectangle, knowing the difference of its base and altitude.*



$$x-y=n \qquad \text{(hypoth.)}$$

$$xy=am \qquad \text{(area of } \triangle\text{)}$$

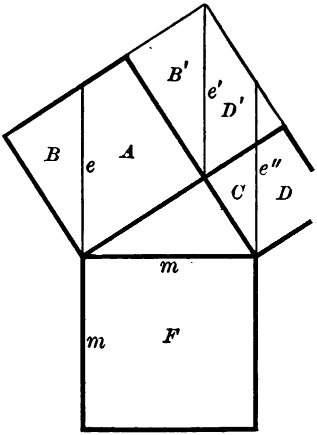
$$\therefore\ x=\frac{n}{2}+\sqrt{am+\left(\frac{n}{2}\right)^2}$$

$$y=-\frac{n}{2}+\sqrt{am+\left(\frac{n}{2}\right)^2}$$

Whence x and y are constructed as shown, the order of the letters indicating the successive steps.

Notice that the *hypothesis* gives one set of equations, the *form of the surface* another; and the whole quest becomes one for the few equations from these two definite sources. The rest is Algebra, until we come to the interpretation.

62. *The square on the hypotenuse is equivalent to the sum of the squares upon the two legs.*



The dominant operation is comparison of surfaces; but before comparison, two of the surfaces must be added, which is performed by transforming the squares into ▱s on the same base, viz.:

$A+B$ into $A+B'$,

$C+D$ into $C+D'$

with the equal bases

$$e=e'=e''=m$$

(△s with two ∠s and a side = etc.)

$$\therefore A+B'+C+D'=F$$

(▱s with = base and alts.)

$$\therefore A+B+C+D=F$$ Q.E.D.

APPENDIX B

SYLLABUS OF PROPOSITIONS.—A HEURISTIC DEVELOPMENT

Axioms: Only one straight line between two points.
A straight line is the shortest.
Coincident magnitudes are =.

One line on the plane gives nothing metrical.

Two lines produce an ∠, suggesting:

1. Vertical ∠s are equal.

The diagram of (1) suggests *symmetry*, and the theorems:

2. Symmetrical lines are =.
3. Symmetrical ∠s are =.
4. Symmetric ∠s on a st. line are rt. ∠s.

A third line produces two new agenda, line segments, and surfaces (limited). By varying the ∠s of this 3-line (triangle) we get, for one rt. ∠:

5. Only one ⊥ from a point to a line.
6. A ⊥ is shorter than any oblique line.

For two rt. ∠s:

7. Lines ⊥ to a third line are ∥.

For a rt. ∠ and a zero ∠:

8. A line ⊥ to one of two ∥s is ⊥ to the other.

For one zero ∠:

9. The alt. int. ∠s of ∥s are =.
10. The ext. int. (corresponding) ∠s of ∥s are =.
11. ∠s with sides ∥ are = or supplementary.
12. ∠s with sides ⊥ are = or supplementary.
13. If alt. int. ∠s are =, the lines are ∥.
14. If the ext. int. ∠s are =, the lines are ∥.
15. If lines are ∥, int. ∠s are supplementary.
16. If int. ∠s are supplementary, the lines are ∥.

Three lines in general:

17. Sum of the ∠s of a △ = 180°.
18. Ext. ∠ of a △ is > opp. int. ∠.
19. Ext. ∠ of a △ = sum of opp. int. ∠s.
20. Sum of two sides of △ is > the third.
21. Difference of two sides of △ is < the third.
22. In a △ the > ∠ is opposite the > side.
23. In a △ the > side is opposite the > ∠.

Pairs of △s:

24. △s with two sides and the incl. ∠ = resp. are =.
25. △s with two ∠s and the incl. side = resp. are =.
26. Two rt. △s with an ∠ and hyp. resp. = are =.
27. Two rt. △s with an ∠ and leg resp. = are =.
28. The base ∠s of an isosceles △ are =.
29. Two rt. △s having a leg and hyp. resp. = are =.
30. An equilateral △ is equiangular.
31. If two ∠s of a △ are =, the opposite sides are =.
32. An equiangular △ is equilateral.
33. Two △s having their sides resp. = are =.
34. The sum of the sides of an interior △ < the sum of the sides of an exterior △ on the same base.
35. If two △s have two sides of the one resp. = to two sides of the other, and the included ∠s unequal, the opposite sides are unequal in the same sense. (Use paper model.)
36. Converse of 35.

Four lines:

37. The opposite sides of a ▱ are =.
38. If the opposite sides of a quadrilateral are =, it is a ▱.
39. If two line segments are = and ∥, they determine a ▱.
40. The diags. of a ▱ bisect each other.
41. The diagonal of a ▱ bisects it.
42. The axis of a kite bisects the diagonal.
43. The axis of a kite is ⊥ to the diagonal.
44. Slope lines cutting equal segments from the foot of the ⊥ are =.
45. The exterior slope line is > the interior slope line.
46. The bisector of the vert. ∠ of an isosceles △ is ⊥ to the base.
47. Slope lines cutting equal segments from the foot of the ⊥ are equally inclined to the ⊥ and to the base line.
48. Equal central angles intercept equal arcs.
49. Equal arcs subtend equal central angles.
50. The > arc subtends the > chord.
51. Equal chords subtend = arcs.
52. The > chord subtends the > arc.
53. A radius ⊥ to a chord bisects it.
54. A ⊥ bisector of a chord passes through the centre.
55. Equal chords are equally distant from the centre.
56. Chords equally distant from the centre are =.
57. The shorter chord is further from the centre.
58. A st. line ⊥ to a radius at its extremity is a tangent.
59. A tangent is ⊥ to the radius to its point of contact.
60. Tangents from a common point are =.
61. Tangents from a common point are equally inclined to the centre line.
62. Central angles have the same ratio as the intercepted arcs.
63. Central angle is measured by the intercepted arc.
64. A radius ⊥ to a chord bisects the subtended arc.
65. Parallel lines intercept = arcs.

66. An inscribed ∠ is measured by ½ the intercepted arc.

67. Parallels cutting equal segments on one transversal cut equal segments on all transversals.

68. A parallel to the base which bisects one side of a △ bisects the other also.

69. A ∥ to the base of a △ cuts the sides proportionally.

70. Mutually equiangular ⟁ are similar.

71. The Pythagorean Proposition.

72. Rectangles with a common altitude are proportional to the bases.

73. Rectangles are to each other as the products of their dimensions.

74. The area of a ▭ = base × alt.

75. The diagonal of a ▱ bisects it.

76. The area of a ▱ = base × alt.

77. The area of a △ = ½ base × alt.

78. (71) The Pythagorean Proposition.

VARIATIONS OF THE ORDER OF DEVELOPMENT

A short cut to computation problems:

$$3, 4, 5, 7, 8, 9, 25, 37, \left\{ \begin{array}{l} 67, 69, 70, 71, \text{ or} \\ 72, 73, 74, 75, 76, 77, 78. \end{array} \right.$$

Milne. 1, 5, 7, 8, 9, 13, 10, 14, 15, 16, 11, 12, 21, 25, 33, 17, 19, 28, 30, 31, 32, 46, 29, 22, 23, 35, 36, 34.

Wentworth. 1, 40, 5, 6, 34, 45, 7, 8, 9, 13, 10, 14, 15, 16, 17, 18, 20, 24, 25, 28, 30, 31, 32, 46, 33, 26, 22, 23, 35, 36, 37, 38, 39, 40.

Schultze and Sevenoak. 1, 25, 21, 18, 13, 14, 16, 9, 10, 15, 11, 17, 19, 28, 30, 31, 32, 33, 29, 42, 43, 20, 21, 33, 23, 35, 36, 37, 38, 39, 40.

Phillips and Fisher. 1, 7, 5, 8, 16, 13, 14, 15, 9, 11, 12, 17, 28, 22, 31, 23, 24, 25, 26, 27, 33, 35, 34, 6, 44, 45, 29, 37, 38, 40, 39.

Wells. 1, 47, 5, 6, 34, 44, conv. of 44, 7, 8, 20, 21, 24, 25, 33, 29, 9, 13, 10, 15, 14, 16, 11, 12, 17, 19, 18, 27, 26, 35, 36, 28, 31, 32, 23, 34, 37, 38, 39, 40.

Beman and Smith. 1, 24, 25, 28, 30, 32, 18, 22, 23, 20, 21, 34, 35, 36, 33, 13, 10, 16, 7, 9, 8.

Holgate. 24, 25, 28, 30, 31, 32, 33, 1, 18, 5, 34, 6, 44, 45, 20, 29.

APPENDIX C

Plane Geometry is the science of those metrical (capable of being measured) continua which are generated by the conjunction of straight lines in a plane, viz. the *angle*, the *line segment*, the *surface segment*, and the ratios existing between elements of the same kind. These four things constitute the sole agenda of plane geometry. The *ultimate aim* of every proposition in geometry is the comparison of pairs of these elements for equality, or inequality, including measurement. This is the *dominant operation.*

Every proposition starts out with a basis of certain "ifs," or premises, at least two. It takes one "if" to provide something to talk about, and at least another "if" to provide some phase as a ground for discussion, in order to ascertain the consequences of the phase. The starting-point for all logical reasoning in geometry is the *axioms*, those truths which are so simple that they need no proof, and so fundamental that they cannot be proved, there being no more fundamental truths upon which to base an argument. From these axioms we build theorems, those truths which generally require proof. Some theorems are so simple that they need no proof, *e.g.* that a straight line cuts a circle twice if at all. But notwithstanding their simplicity, they can be proved, and this is what constitutes them theorems and distinguishes them from axioms.

An important concept in geometry is that of *dimensions*, — those metric properties which are mutually independent of each other, commonly known as length, breadth, and thickness. Any one of these can be changed without affecting the others. It is different with the other geometric elements, side, angle, surface, edge, volume, a change in any one necessitating a change in one or more others.

Following the lead of the ancients, for the same reason that influenced them, elementary plane geometry is restricted to those metrical configurations which can be made by means of the compass and the ungraduated ruler. This limitation is imposed for the reason that we have no other simple instruments for drawing continuous lines. These draw only two kinds, straight and circular. Of these, the straight edge is not strictly an instrument at all, but only a pattern. There is no simple instrument for drawing a straight line. The compass alone is all that is necessary for the constructions of geometry, excepting the one case of drawing a continuous straight line. The compass will determine as many points on a straight line as may be desired, and plant them as thickly as wished; determine a right angle, bisect a given angle, etc., through all the constructions usually made with the aid of a straight edge.

Plane geometry is sharply divided into two parts, the merely metrical part, though it is all metrical, and the mensurational part. The metrical part is that in which the equations which constitute the statements of the subject can be read either as between the lines (or angles), or between the numbers which measure those lines. In this part we utilize only the non-algebraic operations of addition and subtraction, unless we add the comparison of two ratios, and deal with continua. Under this head come the earlier propositions of geometry up to the Pythagorean Theorem in the Third Book. Up to this point the equations are or can be read as equations between lines (or angles) or as proportions between these elements.

The moment, however, that we introduce the Pythagorean Proposition, the square of the hypotenuse, etc., we have crossed the boundary from continua and entered the domain of discreta. Our equations can no longer be expressed in terms of lines, but must be expressed in terms of the numbers measuring lines. For the first time we have introduced the purely algebraic operation of multiplication. We have entered the mensurational part of geometry.

It is well to note that the Pythagorean Theorem expressed as the square *on* the hypotenuse is metrical; expressed as the square *of* the hypotenuse, it is mensurational.

The propositions on the comparative areas of rectangles are metrical, until we reach — the area of a rectangle equals the product of the base by the altitude. This one is mensurational. The elements which enter *must* be numbers. Previously they were surfaces or numbers at will.

In elementary plane geometry we are able to construct and measure only the Euclidean angles, 90°, 60°, 36°, and their multiples, halves, quarters, etc., and combinations of these, *e.g.*:

	90°	45°	22½°	···
120°	60°	30°	15°	···
72°	36°	18°	9°	···
48°	24°	12°	6°	···

Angles not formed of combinations of these[1] cannot be constructed in elementary plane geometry. To draw them we must introduce the scale, which immediately carries us over the line into Trigonometry.

Algebra and geometry are essentially different, but they have a common domain, consisting of the seven equations:

$$x = m \pm n$$

$$x = \frac{lm}{n}$$

$$x = \frac{l^2}{n}$$

$$x = \sqrt{mn}.$$

$$x = \sqrt{m^2 + n^2}.$$

$$x = \sqrt{m^2 - n^2}.$$

---

[1] Omitting angles relating to 17 and 257 sided polygons, as too difficult for elementary geometry.

In these the x can be found whether the other letters represent numbers or lines; x can be constructed without a scale. No other algebraic expressions can be so constructed. Algebraic analysis can be of great service in many geometric investigations, but only when the expressions can be reduced to these seven forms. Some problems in geometry cannot be solved by ruler and compass alone, but must call in the aid of algebraic analysis, and then interpret the result by the aid of these seven forms.

Strictly speaking, the first two are not solely algebraic, in that the symbols can represent things other than numbers: lines, angles, surfaces, sand piles. The third, fourth, and fifth are peculiar in that they can be read as between continua (lines) in ratio form, but *must* be read as between numbers when in any other form. The last two are strictly algebraic, in that the symbols must be numbers. Numbers are the only things which can be multiplied together. What brings these expressions into the domain of geometry is that x can be ascertained by rule and compass alone, notwithstanding the fact that the combinations m^2, mn cannot be represented by geometric elements of the same class as the m and n, viz. lines.

On the contrary, such expressions as $x=lm$, $x=lmn$, $x=\sqrt{lm^2}$, $x=\sqrt[3]{lm}$, $x=\sqrt{m}$ cannot be solved by rule and compass alone, where l, m, n are line segments.

How does it happen that a purely mensurational equation grows out of the metrical ones? Because the properties of similar triangles introduce an equation, $\frac{x}{y}=\frac{m}{n}$, which is metrical in one form, but mensurational in another, $nx=my$. By the combination of two of these mensurational forms comes the Pythagorean equation, which is purely mensurational. Note the three kinds of equations: the purely metrical ones, which can be read as between numbers or lines at will; the semi-mensurational ones, which can be read as between lines or numbers at will in one form, but must be between numbers in another form; and the purely mensurational ones, which must be between numbers only, and which cannot be changed into metrical form at all.

Closely connected with the limitation of the number of algebraic forms capable of being diagrammed is the insolubility of the famous problem of squaring the circle; that is, the construction of a straight line whose length shall equal the circumference; and, as a sequel to this, the finding of its length; that is, a number which shall express its size. Some confusion has arisen at this point, many thinking that the line cannot be constructed by any machine whatever. But this is a mistake. A line equal to the circumference can be drawn by means of a machine called the Integraph, though it cannot be done by rule and compass. The number denoting its length cannot be written, and in this sense the circle cannot be squared.

For many centuries the straight line, the element from which were built the figures under discussion, was unconstructible. It was a curious anomaly, that the figures under discussion were not geometrically capable of being constructed. It was not until 1864 that Peaucellier, a French officer of Engineers discovered how to draw a straight line, and made the first instrument for that purpose. Up to that time all straight lines had been drawn by pattern only; from an easily constructed model used as a guide, the straight edge.

Peaucellier's linkage draws lines of constant curvature, that is circular arcs. By adjustment of the instrument, these arcs can be made to curve either to the right or to the left, or to curve neither way. That is, one phase of these circular arcs is a straight line, a circle whose curvature is zero, whose radius is infinite.

The straight line is often thought of as being the *limit* of the circle, something which the circle tended toward but never quite reached. Peaucellier's linkage shows that the circle does reach the straight line, and that the apparent limit was a result due to the method adopted, the lengthening of the radius. So likewise, one method makes the circle the limit of the circumscribed and inscribed polygons; but another method makes the circle the transition figure between the two sets of polygons, and therefore all theorems applicable to regular polygons likewise applicable to circles, with merely the proper change of language.

MAR 7 1914
MAR 9 1914

Math 5239.05.3
The art of geometry ;
Cabot Science 003339127
3 2044 091 905 950